

Reachability by Paths of Bounded Curvature in a Convex Polygon^{*}

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Abstract

Let B be a point robot moving in the plane, whose path is constrained to forward motions with curvature at most one, and let P be a convex polygon with n vertices. Given a starting *configuration* (a location and a direction of travel) for B inside P , we characterize the region of all points of P that can be reached by B , and show that it has complexity $O(n)$. We give an $O(n^2)$ time algorithm to compute this region. We show that a point is reachable only if it can be reached by a path of type CCSCS, where C denotes a unit circle arc and S denotes a line segment.

1 Introduction

The problem of planning the motion of a robot subject to non-holonomic constraints [16, 25] (for instance, bounds on velocity or acceleration [9, 11, 22], bounds on the turning angle) has received considerable attention in the robotics literature. Theoretical studies of non-holonomic motion planning are far sparser.

In this paper we consider a point robot in the plane whose turning radius is constrained to be at least one and that is not allowed to make reversals. This restriction corresponds naturally to constraints imposed by the steering mechanism found in car-like robots. We assume that the robot is located at a given position (and orientation) inside a convex polygon, and we are interested in the set of points in the polygon that can be reached by the robot. We put no restriction on the orientation with which the robot can reach a point.

The lack of such a restriction distinguishes our work from most of the previous theoretical work on curvature-constrained paths, which usually assumes that not a point, but a *configuration* (a location with orientation) is given. Dubins [12] was perhaps the first to study curvature-constrained shortest paths. He proved that a curvature-constrained shortest path from a given starting configuration to a given final configuration consists of at most three segments, each of which is either a straight line or an arc of a unit-radius circle. Reeds and Shepp [21] extended this characterization to robots that are allowed to make reversals. Using ideas from control theory, Boissonnat et al. [6] gave an alternative proof for both cases, and Sussmann [26] extended the characterization to the 3-dimensional case.

In the presence of obstacles, Fortune and Wilfong [13] gave a single-exponential decision procedure to verify if two given configurations can be joined by a curvature-constrained path avoiding the polygonal obstacles. On the other hand, computing a *shortest* bounded-curvature path among polygonal obstacles is NP-hard, as shown by Reif and Wang [23]. Wilfong [27] designed an exact algorithm for the case where the curvature-constrained path is limited to some fixed straight “lanes” and circular arc turns between the lanes. Agarwal et al. [2] considered the case of disjoint convex obstacles whose curvature is also bounded by one, and gave efficient approximation algorithms. Boissonnat and Lazard [8] gave a polynomial-time algorithm for computing the exact shortest paths for the case when the edges of the

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obstacles are circular arcs of unit radius and straight line segments. Boissonnat et al. [7] gave an $O(n^4)$ algorithm for finding a *convex and simple* path of bounded curvature within a *simple polygon*. Agarwal et al. [1] presented an $O(n^2 \log n)$ -time algorithm to compute a curvature-constrained shortest path between two given configurations inside a *convex polygon*. They also showed that there exists an optimal path that consists of at most eight line segments or circular arcs. For general polygonal obstacles, Backer and Kirkpatrick [5] recently gave the first complete approximation algorithm, improving on earlier work that approximated the shortest “robust” path [15, 3].

At least two interesting problems have been studied where not configurations but only locations for the robot are given. The first problem considers a *sequence* of points in the plane, and asks for the shortest curvature-constrained path that visits the points in this sequence. In the second problem, the *Dubins traveling salesman problem*, the input is a *set* of points in the plane, and asks to find a shortest curvature-constrained path visiting all points. Both problems have been studied by researchers in the robotics community, giving heuristics and experimental results [24, 18, 19]. From a theoretical perspective, Lee et al. [17] gave a linear-time, constant-factor approximation algorithm for the first problem. No approximation algorithms are known for the Dubins traveling salesman problem.

Our result is a characterization of the region of points reachable by paths under curvature constraints from a given starting configuration inside a convex polygon P . We show that all points reachable from the starting configuration are also reachable by paths of type CCSCS, where C denotes an arc of a unit-radius circle and S denotes a line segment. When P has n vertices, we show that the reachable region has complexity $O(n)$, and we give an $O(n^2)$ time algorithm to compute this region.

2 Terminology and some lemmas

Let P be a convex polygon in the plane. A *configuration* $\mathbf{s} = (s, \mathbf{d})$ is a point s together with a direction of travel \mathbf{d} (a unit vector). By a *path*, we mean a continuously differentiable curve (the image of a C^1 -mapping of $[0, 1]$ to \mathbb{R}^2) with average curvature bounded by one in every positive-length interval. Unless stated otherwise, we assume that a path is completely contained in P . A *configuration on the path* π is a configuration $\mathbf{s} = (s, \mathbf{d})$ with s on π such that \mathbf{d} is the forward tangent to π in s . The *starting configuration* of π is the starting point of π with its forward tangent. A *simple path* is a path with no self-intersection; we allow the endpoints of a simple path to coincide, in which case we call it a *simple closed path*. (Hence, a simple closed path is smooth except possibly at one point.)

Given a configuration $\mathbf{s} = (s, \mathbf{d})$, the *left disk* $D_L(\mathbf{s})$ (*right disk* $D_R(\mathbf{s})$) is the unit disk touching s and completely contained in the left (right) halfplane defined by the directed line through \mathbf{d} . (All *unit disks* in this paper are *unit-radius* disks.)

The *left directly accessible region* $\text{LDA}(\mathbf{s})$ is the set of all points in P that can be reached by a path with starting configuration \mathbf{s} consisting of a single (possible zero-length) circular arc on the boundary of $D_L(\mathbf{s})$ followed by a single (possible zero-length) line segment. The *right directly accessible region* is defined analogously. The *directly accessible region* $\text{DA}(\mathbf{s})$ is the union of the left and right directly accessible region.

Pestov-Ionin lemma. The following lemma is perhaps the foundation for all our results. In a slightly less general form, it was proven by Pestov and Ionin [20]. Recently, it has been used for a curve reconstruction problem [14].

Lemma 1 (Pestov-Ionin). *Any simple closed path contains a unit disk in its interior.*

For sake of completeness, we sketch a proof analogous to Pestov and Ionin’s.

Lemma 2. *Let D be a closed disk, and Γ be a simple path with endpoints (a, b) such that $\Gamma \cap D = \{a, b\}$. Then there is a unit disk touching $\Gamma \setminus \{a, b\}$ that lies within the region \mathcal{R} bounded by Γ and the exterior arc of ∂D (See Figure 1).*

Proof. We proceed by induction on the length ℓ of Γ (or, more precisely, by induction on $\lfloor \ell/\pi \rfloor$).

When $\ell < \pi$, we can prove by integration that a unit disk tangent to Γ does not cross Γ . We consider a unit disk D_0 tangent to Γ at $m \notin \{a, b\}$ on the interior side. Since $D_0 \setminus D$ has only one connected component and $m \in D_0 \setminus D$, clearly D_0 is contained in \mathcal{R} .

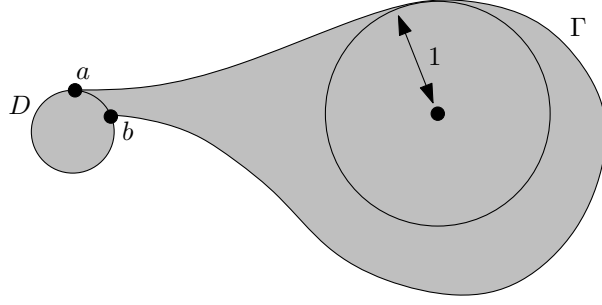


Figure 1: Illustration of Lemma 2. The region \mathcal{R} is shaded.

Otherwise, let m be the point halfway between a and b on Γ . Let D_1 be the largest disk contained in \mathcal{R} that is tangent to Γ in m . If the radius of D_1 is larger or equal to 1, then we are done. If not, D_1 must touch $\partial\mathcal{R}$ in another point m' . Clearly m' lies on Γ , and the length of the arc Γ' of Γ between m and m' is at most $\ell/2$. By the induction hypothesis, a unit disk D_2 lies inside the region \mathcal{R}_1 bounded by Γ' and D_1 . Since $\mathcal{R}_1 \subseteq \mathcal{R}$, the lemma follows. \square

Lemma 1 follows from Lemma 2 by observing that it still holds when D degenerates to a point.

Filling. By $\text{FIL}(P)$ we denote the *set* of all unit-radius disks that are completely contained in P , and we let $\text{Fil}(P)$ be the union of all the disks in $\text{FIL}(P)$. (See Figure 2.) Both $\text{FIL}(P)$ and $\text{Fil}(P)$ will be called the *filling* of P .

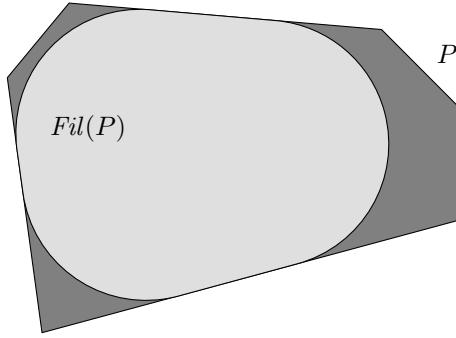


Figure 2: The filling $\text{Fil}(P)$ is shaded in light grey, and the three pockets are shaded in dark grey.

Pockets. The connected components of $P \setminus \text{Fil}(P)$ are called the *pockets* of P . A pocket of P is bounded by a single circular arc (lying on one disk of $\text{FIL}(P)$) and a connected part of the boundary of P . The first and last edge on this connected chain are called the *mouth edges* of the pocket. Its extremities are called the *mouth points*. The mouth edges form an angle smaller than π (this is equivalent to observing that the mouth points lie on the same disk of $\text{FIL}(P)$ and form an angle smaller than π) [1]. Agarwal et al. [1] proved the following lemma.

Lemma 3 (Pocket lemma). *A path entering a pocket from $\text{Fil}(P)$ cannot leave the pocket anymore.*

Proof. We consider a pocket K bounded by a disk D . Suppose that there is a path Γ that enters and leaves K . There is a subpath Γ' of Γ whose endpoints lie on D and whose other points are in the interior of K . By Lemma 2, there is a unit disk D' that touches Γ' at a point in the interior of K , and is contained in $K \cup D$. Hence, $D' \in \text{FIL}(P)$, and D' intersects the interior of K , a contradiction. \square

Reachability for a union of disks. For a set \mathcal{D} of unit disks, we let $\text{CONV}(\mathcal{D})$ denote the set of all unit disks contained in the convex hull of $\bigcup \mathcal{D}$. Equivalently, $\text{CONV}(\mathcal{D})$ consists of the unit disks centered at points of the convex hull of the set of all centers of the disks in \mathcal{D} .

Observation 4. *Let \mathcal{D} be a set of unit disks in the plane. Then $\bigcap \mathcal{D} = \bigcap \text{CONV}(\mathcal{D})$.*

Given a convex set Q , a *configuration on the boundary* of Q is a configuration $\mathbf{s} = (s, \mathbf{d})$ with s on the boundary of Q and such that \mathbf{d} is tangent to the boundary of Q in s .

Lemma 5. *Let \mathcal{D} be a set of unit disks, and let \mathbf{s} be a starting configuration on the boundary of $\bigcup \text{CONV}(\mathcal{D})$. Then no point in the interior of $\bigcap \mathcal{D}$ can be reached by a path starting at \mathbf{s} and contained in $\bigcup \text{CONV}(\mathcal{D})$, or even in any convex polygon P such that $\text{FIL}(P) = \text{CONV}(\mathcal{D})$.*

Proof. Assume to the contrary that there is a path γ with starting configuration \mathbf{s} on the boundary of $\bigcup \text{CONV}(\mathcal{D})$ and ending point t in the interior of $\bigcap \mathcal{D}$. Assume for the moment that γ lies completely in $\bigcup \text{CONV}(\mathcal{D})$.

We extend γ to infinity using a straight ray, such that the extended path is still C^1 . We then extend γ backwards, by attaching a single loop around the boundary of $\bigcup \text{CONV}(\mathcal{D})$ at \mathbf{s} . To summarize, the extended path γ' starts at \mathbf{s} , makes a single loop around the boundary of $\bigcup \text{CONV}(\mathcal{D})$, then follows the original path γ , and finally escapes to infinity along a straight line. We can now construct a simple, closed path γ'' as follows: starting at infinity, we follow γ' *backwards*, until we encounter the first intersection of γ' with the part that we have already seen. Such an intersection must exist since the two extensions

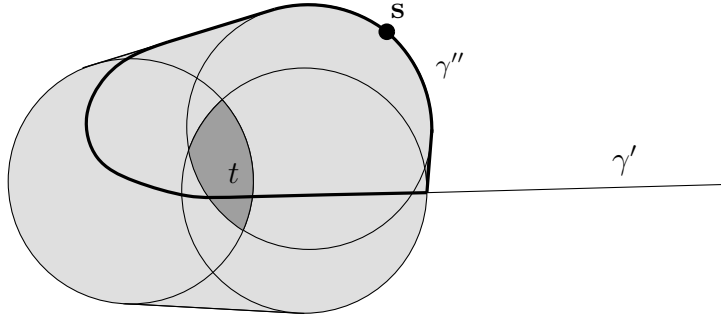


Figure 3: Proof of Lemma 5. The set \mathcal{D} consists of three disks. The convex hull $\text{CONV}(\mathcal{D})$ is shaded in light grey, and $\bigcap \mathcal{D}$ is in dark grey. The path γ from s to t is enlarged into the path γ'' .

intersect. We define γ'' to be the part of γ' between these two self-intersection points. Observe that t lies either on or outside the closed loop γ'' .

By the Pestov-Ionin Lemma, γ'' contains a unit-disk D . Since γ'' is contained in $\bigcup \text{CONV}(\mathcal{D})$, we have $D \in \text{CONV}(\mathcal{D})$. Consequently, the interior of $\bigcap \mathcal{D}$ lies in the interior of D , a contradiction with the fact that t must lie on or outside γ'' .

The lemma still holds when we allow the path to lie inside a convex polygon P with $\text{FIL}(P) = \text{CONV}(\mathcal{D})$. After all, by Lemma 3, the path cannot return to $\text{Fil}(P)$ after it has left it. \square

The characterization. Consider a starting configuration \mathbf{s} on the boundary of the filling $\text{Fil}(P)$. It is easy to see that any point in P not in $\bigcap \text{FIL}(P)$ can be reached by a path from \mathbf{s} . On the other hand, by Lemma 5, no point in $\bigcap \text{FIL}(P)$ can be reached, and so we have a complete characterization of the region reachable from \mathbf{s} as the complement of $\bigcap \text{FIL}(P)$.

If the starting configuration lies on the boundary of an arbitrary unit disk contained in P , the same characterization holds. For arbitrary starting configurations, however, the situation becomes far more complicated. There is the possibility that no path starting at \mathbf{s} is tangent to the boundary of $\text{Fil}(P)$, and the filling has no relation to the reachable region.

If there exists a path starting at \mathbf{s} that is tangent to the boundary, all points outside $\bigcap \text{FIL}(P)$ are reachable, but it is still possible that some points inside $\bigcap \text{FIL}(P)$ are reachable, for instance because they lie in the directly reachable area $\text{DA}(\mathbf{s}) = \text{LDA}(\mathbf{s}) \cup \text{RDA}(\mathbf{s})$, or because \mathbf{s} lies in a pocket with additional maneuvering space (that we would not have been able to exploit if starting inside $\text{Fil}(P)$ by Lemma 3). (See Figure 4).

In the rest of this paper, we give a complete characterization of the reachable region, for any starting configuration in P . Let us denote the set of points $t \in P$ such that t is reachable by a path starting from a configuration \mathbf{s} by $\text{REACH}(\mathbf{s})$.

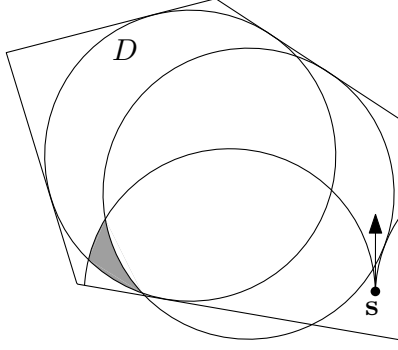


Figure 4: Points in the grey region are reachable from \mathbf{s} , but are neither in $DA(\mathbf{s})$ nor in the complement of $D = \bigcap \text{FIL}(P)$.

3 Paths starting along the boundary

In this section, we assume that the starting configuration \mathbf{s} is on the boundary of P (recall that this means also that the direction is tangent to the boundary). Without loss of generality, we also assume that the direction of \mathbf{s} is counterclockwise along the boundary, so that points of P are reached locally by a left turn from \mathbf{s} . It turns out that in this situation we can restrict ourselves to paths containing no right-turning arcs.

The *forward chain* $FC(\mathbf{s})$ is the longest subchain of the boundary of P , that starts counterclockwise from \mathbf{s} , and that turns by an angle at most π . (See Figure 5.) In other words, when \mathbf{s} is directed

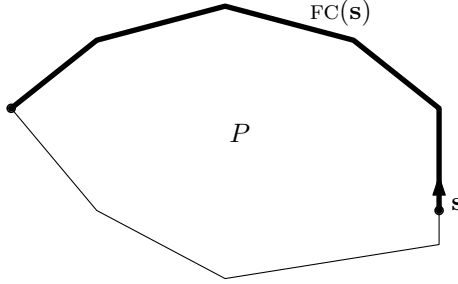


Figure 5: The forward chain $FC(\mathbf{s})$.

vertically upward, this chain contains all the edges of P that are above its interior, as well as the part of the edge that contains s and is above s , and, if there is one, the other vertical edge of P .

If the forward chain intersects the interior of $D_L(\mathbf{s})$, then we have the following simple description of the reachable region.

Lemma 6. *If the forward chain $FC(\mathbf{s})$ intersects the interior of $D_L(\mathbf{s})$, then $\text{REACH}(\mathbf{s}) = \text{LDA}(\mathbf{s})$.*

Proof. The left directly accessible region $\text{LDA}(\mathbf{s})$ can be enlarged to a pocket of the left disk $D_L(\mathbf{s})$. Thus, no point outside $\text{LDA}(\mathbf{s})$ is reachable by the Pocket Lemma (Lemma 3). \square

We denote by $\text{LFIL}(\mathbf{s})$ the set of disks contained in $P \cup D_L(\mathbf{s})$ that touch the forward chain. Note that $\text{LFIL}(\mathbf{s})$ always contains the left disk $D_L(\mathbf{s})$. Let us remark that the set of centers of the disks in $\text{LFIL}(\mathbf{s})$ need not be connected. For example, Figure 6 shows a situation where $\text{LFIL}(\mathbf{s})$ consists of just three disks; their centers are marked by black dots (in general, any number of connected components is possible). For a disk $F \in \text{LFIL}(\mathbf{s})$, we define $\text{LDA}(F)$ as $\text{LDA}(\mathbf{r})$, where \mathbf{r} is the first configuration on $FC(\mathbf{s})$ touching F (by the above, this is well defined). Note that if $F \in \text{FIL}(P)$, then $\text{LDA}(F)$ is simply the complement of the interior of F . We continue with a lemma on the reachable points outside $\text{LDA}(\mathbf{s})$.

Lemma 7. *Let P and \mathbf{s} be as above, with $D_L(\mathbf{s}) \notin \text{FIL}(P)$, and the forward chain $FC(\mathbf{s})$ not intersecting the interior of $D_L(\mathbf{s})$. Suppose that there is a path γ from \mathbf{s} to t , where $t \notin \text{LDA}(\mathbf{s})$. Then there exists a disk F such that $F \in \text{FIL}(P)$ and t is not in the interior of F , or there exists a disk $F \in \text{LFIL}(\mathbf{s})$ such that $t \in \text{LDA}(F)$.*

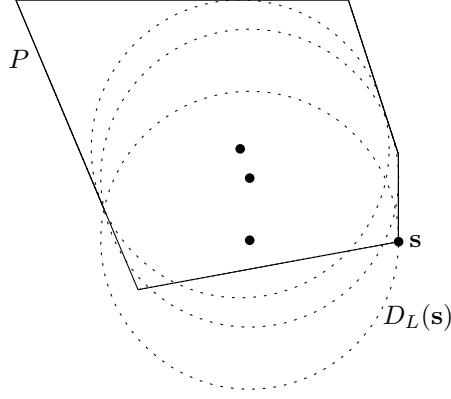


Figure 6: Example where the left filling consists of just 3 disks.

Proof. Without loss of generality, we assume that s is directed vertically upward. We first assume that t is in the interior of $D_L(s)$.

Consider the line st . Let us first assume that γ intersects this line top-to-bottom (or tangentially) at t . In that case, we extend γ forward by the semi-infinite ray starting at t , and backward by the boundary of $D_L(s)$. We trace the resulting path backwards from infinity, stopping at the first intersection of the path with the part we have already seen, and thus forming a loop that does not contain t in its interior. We apply the Pestov-Ionin lemma to this loop, and find a unit-radius disk D contained in it. (See Figure 7.) If $D \in \text{FIL}(P)$, then we are done. Otherwise, note that the loop does not cross the boundary

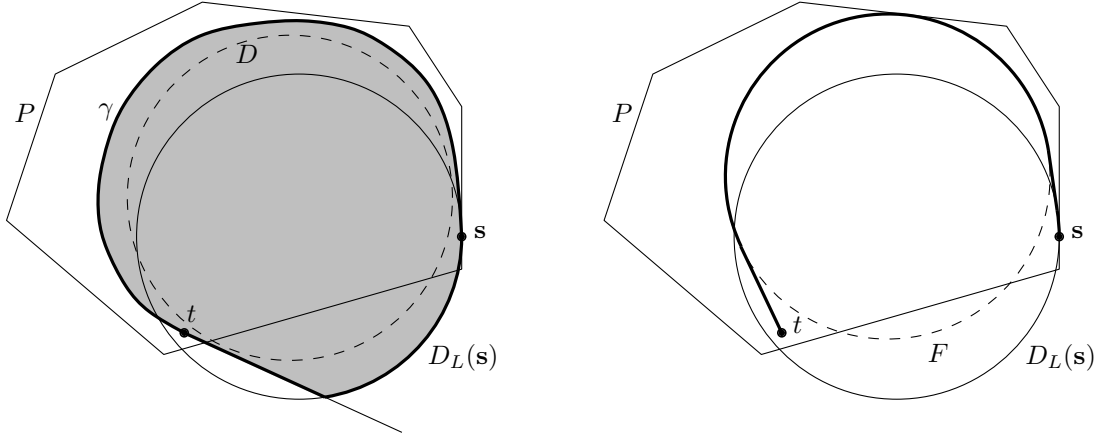


Figure 7: (left) A unit disk D lies in the shaded area. (right) t is in $\text{LDA}(F)$.

of $P \cup D_L(s)$, so $D \in P \cup D_L(s)$. Then we obtain a disk $F \in \text{LFIL}(s)$ by translating D upwards until it touches the forward chain $\text{FC}(s)$, and we have $t \in \text{LDA}(F)$.

Now we consider the case where γ intersects the line st bottom-to-top at t . Let t' be the first point of intersection of γ and the line st (along the path γ). If t' lies between s and t on the line st , then we can apply the argument above to conclude the existence of a disk F such that $t' \in \text{LDA}(F)$ —but then also $t \in \text{LDA}(F)$.

If, finally, t lies between s and t' on st , then we extend γ by the semi-infinite ray starting in t . This ray must intersect the part of γ from s to t' , and so again we have found a loop lying in P , fulfilling the requirements of the Pestov-Ionin lemma, and containing t . As above, there is then a disk $F \in \text{FIL}(P)$ with $t \in \text{LDA}(F)$.

We now consider the case where t does not lie in the interior of $D_L(s)$. Then t lies in a connected component P_c of $P \setminus D_L(s)$ different from $\text{LDA}(s)$. Since $D_L(s)$ does not intersect the forward chain, P_c lies entirely below $D_L(s)$. Let t' denote the first point on γ that is on the boundary of P_c . We trace backward from t' a path along γ , and then along the lower semi-circle of $D_L(s)$, until we reach a point that we have already seen. It forms a loop on which we apply the Pestov-Ionin lemma. Thus we find a

disk D inside $P \cup D_L(\mathbf{s})$ that does not contain $\{t, t'\}$. If $D \in \text{FIL}(P)$, then we choose $F = D$ and we are done. Otherwise, we obtain F by translating D upward until it meets the forward chain, and we have $t \in \text{LDA}(F)$. \square

The lemma above does not give us a complete characterization of the reachable region, as we do not know yet whether we can reach the disks in $\text{FIL}(P)$ and $\text{LFIL}(P)$ tangentially. The following lemma addresses this issue.

Lemma 8. *Assume that the forward chain $\text{FC}(\mathbf{s})$ does not intersect the interior of $D_L(\mathbf{s})$. (i) If $F \in \text{FIL}(P)$, then there exists a counterclockwise configuration tangent to F that can be reached from \mathbf{s} by a CS path. (ii) If $F \in \text{LFIL}(P)$, then the first configuration on $\text{FC}(\mathbf{s})$ tangent to F can be reached from \mathbf{s} by a CSC path.*

Proof. The lemma is obvious when F touches the edge containing s , so we assume this is not the case. We first prove (i). If we draw a line segment upward from the leftmost point p of F until we meet the forward chain, we do not intersect $D_L(\mathbf{s})$. It follows that $p \in \text{LDA}(\mathbf{s})$. We consider a ray that starts at a configuration \mathbf{s}' tangent to $D_L(\mathbf{s})$. We start at $\mathbf{s}' = \mathbf{s}$ and move \mathbf{s}' counterclockwise along the boundary of $D_L(\mathbf{s})$, so that the ray sweeps $\text{LDA}(\mathbf{s})$. Since $p \in \text{LDA}(\mathbf{s})$, this ray must meet F at some point. When the ray first meets F , it is tangent to F , so we can reach the corresponding configuration by a CS path.

We now prove (ii). We denote by $\mathbf{r} = (r, \mathbf{d})$ the first configuration on the forward chain that is tangent to F . Then $r \in \text{LDA}(\mathbf{s})$, so when we sweep the same ray as in the proof of (i), we meet F tangentially at some configuration \mathbf{r}' . The arc between \mathbf{r}' and \mathbf{r} is inside P , so we have a CSC path starting from \mathbf{s} and going through \mathbf{r}' and \mathbf{r} . \square

We are now able to give the following characterization for the reachable region starting from a configuration on the side of P . It follows directly from Lemmas 6, 7, and 8.

Proposition 9. *Assume that \mathbf{s} is a configuration on the boundary of P , oriented counterclockwise. Then any point in $\text{REACH}(\mathbf{s})$ can be reached by a CSCS path. In addition, we have that:*

(i) *If $\text{FC}(\mathbf{s})$ intersects the interior of $D_L(\mathbf{s})$, then $\text{REACH}(\mathbf{s}) = \text{LDA}(\mathbf{s})$.*

(ii) *If $\text{FC}(\mathbf{s})$ does not intersect the interior of $D_L(\mathbf{s})$, then $\text{REACH}(\mathbf{s}) = \bigcup_{F \in \text{FIL}(P) \cup \text{LFIL}(\mathbf{s})} \text{LDA}(F)$.*

In the characterization above, it seems that an infinite number of disks could possibly contribute to the boundary of the reachable region. In the following, we show that the contribution of the LDAs along any edge can be reduced to at most two LDAs. We focus on a particular edge f . Let \mathbf{d} denote the counterclockwise direction along this edge. Then we order the counterclockwise configurations along f according to direction \mathbf{d} , that is, for two such configurations $\mathbf{s}_1 = (s_1, \mathbf{d})$ and $\mathbf{s}_2 = (s_2, \mathbf{d})$, we say that $\mathbf{s}_1 \preceq \mathbf{s}_2$ when $\overrightarrow{s_1 s_2} \cdot \mathbf{d} \geq 0$.

Lemma 10. *Let $\mathbf{s}_1 = (s_1, \mathbf{d})$ and $\mathbf{s}_2 = (s_2, \mathbf{d})$ be two counterclockwise configurations on the same edge f of P , such that $\mathbf{s}_1 \preceq \mathbf{s}_2$. Let $\mathbf{h}_1 = (h_1, \mathbf{d})$ denote the first counterclockwise configuration on f such that $\mathbf{s}_1 \preceq \mathbf{h}_1$ and $D_L(\mathbf{h}_1)$ intersects $\text{FC}(\mathbf{h}_1) \setminus \{h_1\}$. Then $\text{LDA}(\mathbf{s}_2) \subset \text{LDA}(\mathbf{s}_1) \cup \text{LDA}(\mathbf{h}_1)$.*

Proof. We first assume that $\text{FC}(\mathbf{s}_1)$ intersects the interior of $D_L(\mathbf{s}_1)$, so that $\mathbf{h}_1 = \mathbf{s}_1$. Then $\text{LDA}(\mathbf{s}_1)$ can be enlarged into a pocket, so by Lemma 3, we have $\text{LDA}(\mathbf{s}_2) \subset \text{LDA}(\mathbf{s}_1)$.

Otherwise, $\text{FC}(\mathbf{s}_1)$ does not intersect the interior of $D_L(\mathbf{s}_1)$. Thus, the disk $D_L(\mathbf{h}_1)$ touches $\text{FC}(\mathbf{s}_1)$. (See Figure 8.) If $\mathbf{h}_1 \preceq \mathbf{s}_2$, then $\text{LDA}(\mathbf{h}_1)$ is a pocket, so by Lemma 3, we have $\text{LDA}(\mathbf{s}_2) \subset \text{LDA}(\mathbf{h}_1)$.

Finally, we assume that $\mathbf{s}_1 \preceq \mathbf{s}_2 \preceq \mathbf{h}_1$. Let t denote a point in $\text{LDA}(\mathbf{s}_2)$. If t is reached after an arc of $D_L(\mathbf{s}_2)$ with length less than π followed by a line segment $r_2 t$, then it is clearly in $\text{LDA}(\mathbf{s}_1)$. (See Figure 8, left.) On the other hand, if t is reached by an arc of $D_L(\mathbf{s}_2)$ with length at least π , followed by a segment $r_2 t$, we claim that $t \in \text{LDA}(\mathbf{h}_1)$. Let r be the point of $D_L(\mathbf{h}_1)$ such that the line rt is tangent to $D_L(\mathbf{h}_1)$ from the left. (See Figure 8, right.) We have to argue that the arc of $D_L(\mathbf{h}_1)$ from h_1 to r lies in P . This follows from the fact that $D_L(\mathbf{h}_1)$ is obtained by translating $D_L(\mathbf{s}_2)$ along f until it touches $\text{FC}(\mathbf{s}_1)$, that the arc of $D_L(\mathbf{s}_2)$ from s_2 to r is in P , and that the arc from h_1 to r_2 is shorter than the arc from s_2 to r . \square

Now we can show how to construct the reachable region from a configuration on the boundary.

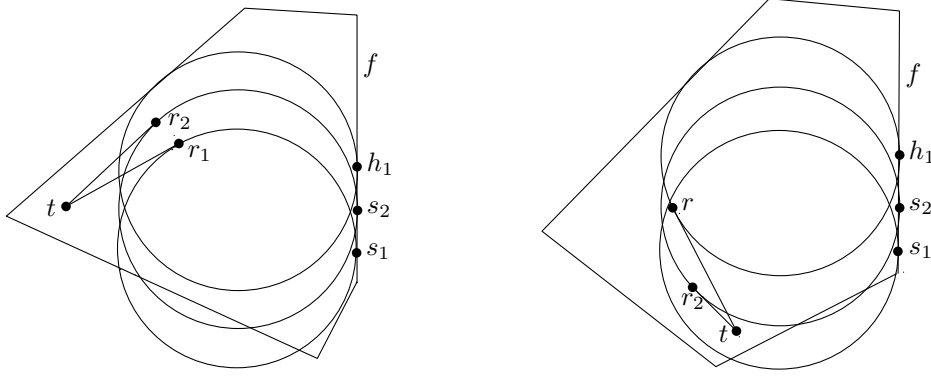


Figure 8: Proof of Lemma 10.

Proposition 11. *Let \mathbf{s} be a counterclockwise configuration on the boundary of an n -sided convex polygon P . Then we can compute in $O(n)$ time a set $\text{BFIL}(\mathbf{s})$ of $O(n)$ disks, such that $\text{REACH}(\mathbf{s}) = \bigcup_{F \in \text{BFIL}(\mathbf{s})} \text{LDA}(F)$.*

Proof. We use the characterization of $\text{REACH}(\mathbf{s})$ from Proposition 9. If $\text{FC}(\mathbf{s})$ intersects the interior of $D_L(\mathbf{s})$, then we just set $\text{BFIL}(\mathbf{s}) = \{D_L(\mathbf{s})\}$. So in the remainder of this proof, we assume that $\text{FC}(\mathbf{s})$ does not intersect the interior of $D_L(\mathbf{s})$.

If $\text{FIL}(P) \neq \emptyset$, then we first construct the contribution of the disks in $\text{FIL}(P)$. By Observation 4, we only need to find the unit disks whose centers are the vertices of the convex hull of the centers of the disks in $\text{FIL}(P)$. These disks are tangent to at least two edges of P , so their centers lie on the medial axis [4, 10] of P . We compute this medial axis in $O(n)$ time using an algorithm by Aggarwal et al. [4], and then check each edge on the medial axis to obtain these disks in $O(n)$ time.

We now observe that if $D_L(\mathbf{s}) \in \text{FIL}(P)$, then we can set $\text{BFIL}(\mathbf{s}) = \text{FIL}(\mathbf{s})$ and are done. Thus, in the remainder of this proof, we assume that $D_L(\mathbf{s}) \notin \text{FIL}(P)$, and explain how to find the contribution of $\text{LFIL}(P)$.

Let d denote the first point on ∂P , starting from s in counterclockwise direction, such that $d \in D_L(\mathbf{s})$. (See Figure 9(left).) Let us call a *candidate configuration* a configuration \mathbf{s}' on the counterclockwise boundary of P with the property that $D_L(\mathbf{s}') \subset P \cup D_L(\mathbf{s})$ and such that $D_L(\mathbf{s}')$ is either tangent to two edges of $\text{FC}(\mathbf{s})$, or is tangent to one edge of $\text{FC}(\mathbf{s})$ and contains the point d .

Consider an arbitrary edge f of P . Let \mathbf{s}_1 denote the first counterclockwise configuration on f such that $D_L(\mathbf{s}_1) \in \text{LFIL}(P)$. When \mathbf{h}_1 is as in Lemma 10, the LDAs of all the disks in $\text{LFIL}(P)$ that are tangent to f are contained in $\text{LDA}(\mathbf{s}_1) \cup \text{LDA}(\mathbf{h}_1)$. So we only need to find \mathbf{s}_1 and \mathbf{h}_1 to construct the contribution of f to $\text{REACH}(\mathbf{s})$.

We observe that \mathbf{s}_1 and \mathbf{h}_1 are candidate configurations: in fact, \mathbf{s}_1 is the *first* candidate configuration on f , while \mathbf{h}_1 is the *last*.

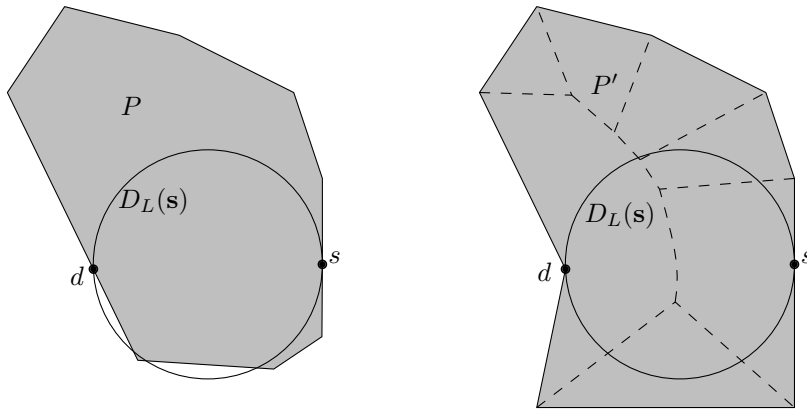


Figure 9: On the left, P is shaded. On the right, P' is shaded, and its medial axis is dashed.

It remains to explain how to compute the candidate configurations efficiently. Denote by P' a simple polygon obtained by replacing the subchain of ∂P that goes counterclockwise from d to s with two or three edges, such that $D_L(s) \subset P'$. (See Figure 9(right).) The disk $D_L(s')$, for a candidate configuration s' , must lie in P' and must touch $\partial P'$ in more than one point. It follows that we can find the candidate configurations by first computing the medial axis of P' in $O(n)$ time using the algorithm by Chin et al. [10], and then checking all edges of the medial axis. \square

Proposition 11 shows that the reachable region for a configuration on the boundary of P is delimited by $O(n)$ disks. This bound is tight, as shown by the example in Figure 10, where $\Omega(n)$ disks of $\text{LFIL}(s)$ contribute to the boundary of the reachable region.

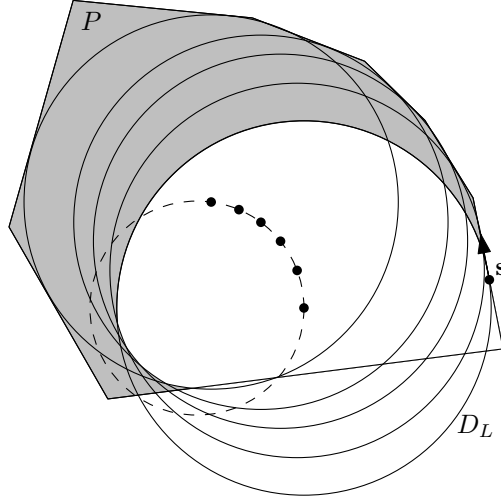


Figure 10: Example where $\Omega(n)$ disks of $\text{LFIL}(s)$ contribute to the boundary of the reachable region. Here the disks of $\text{LFIL}(s)$ are centered along the dashed circle. The reachable region is shaded.

4 Special left-right and right-left paths suffice

Let s be a starting configuration. A *canonical RL-start* from s is a path from s to a configuration r on the boundary of P that begins with a right-turning arc of unit radius and continues with a left-turning arc of unit radius ending at r (and tangent to the boundary of P there) (Figure 11).

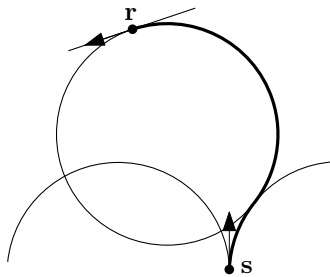


Figure 11: A canonical RL-start.

Note that for each edge f of P and for a given s , there are at most two canonical RL-starts from s ending on f . A canonical LR-start is defined analogously: it begins with a left-turning arc and continues with a right-turning arc.

In this section, we show that for determining the reachability by paths in a convex polygon, it suffices to consider paths of a fairly special form. Namely, we show that a point is reachable if and only if it is directly accessible, or it can be reached by a path that begins with a canonical start.

Dubins [12] showed that the shortest path of bounded curvature between two configurations in the plane is of type CSC or CCC. In the latter case, the middle arc has length more than π . (See Figure 12.) We call these paths *Dubins paths*.

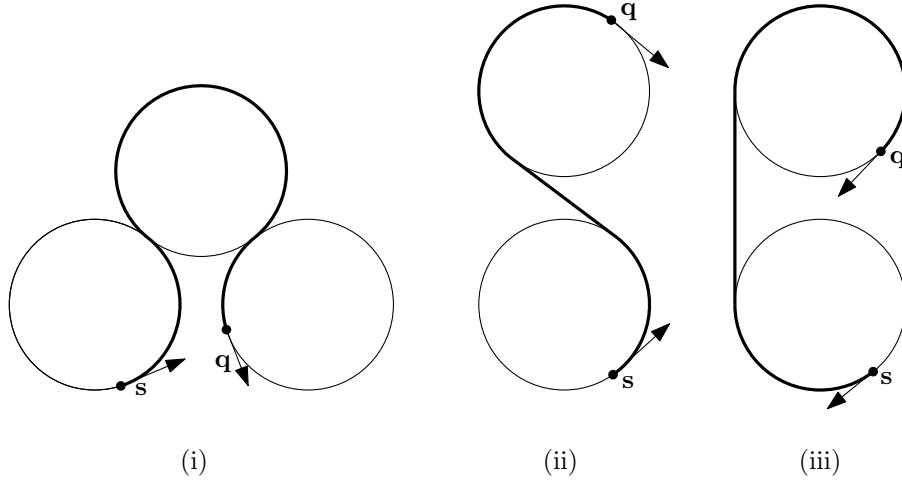


Figure 12: Three types of Dubins paths.

Proposition 12. *Let P be a convex polygon, let s be a starting configuration in P , and let $t \in P$ be reachable from s by a bounded-curvature path. Then t lies in the directly accessible region $DA(s)$, or it can be reached by a path of one of the following forms: a canonical RL -start followed by a left-turning path (starting on a side of P), or a canonical LR -start followed by a right-turning path (starting on a side of P).*

Proof. Jacobs and Canny [15] showed that, in a polygonal environment, the shortest path of bounded curvature between two configurations is a sequence of Dubins paths. The final configurations of these Dubins paths (except for the last one) all lie on the boundary of the polygonal environment. So, if we denote by γ a shortest path from s to t , then γ can be written as a sequence $\gamma = \gamma_1 \gamma_2 \dots \gamma_m$ of m Dubins paths. When $m \geq 2$, we know that the final configuration q of γ_1 lies on ∂P .

We handle three cases separately, according to the type of γ_1 (see Figure 12). If $D_L(s)$ and $D_R(s)$ are contained in P , then $DA(s) = P$, so from now on, we assume that $D_L(s)$ or $D_R(s)$ crosses ∂P .

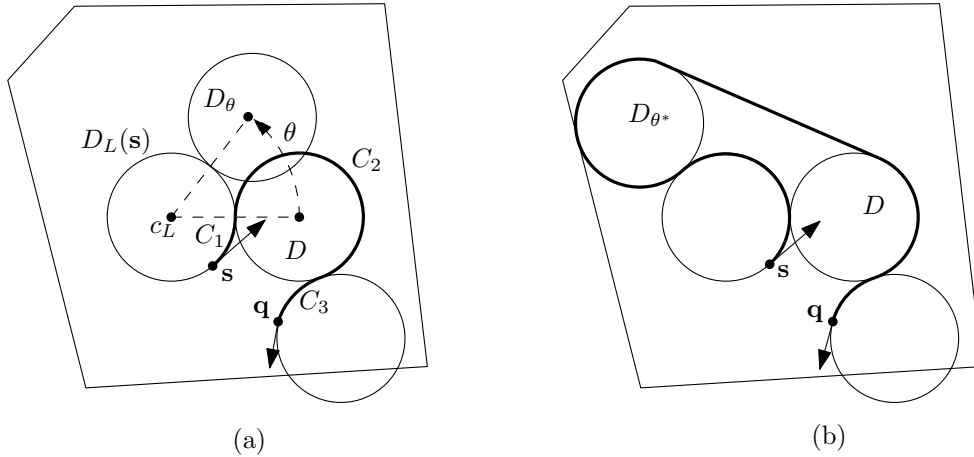


Figure 13: Proof of Proposition 12, case (i).

Case (i). We assume that γ_1 is of type CCC . We denote by C_1, C_2, C_3 the three circle arcs such that $\gamma_1 = C_1 C_2 C_3$, and recall that C_2 has length larger than π . If C_2 touches ∂P we are done, so from now

on we assume that C_2 does not touch ∂P . Let D be the disk supporting C_2 . Without loss of generality, we assume that C_1 and C_3 turn counterclockwise and C_2 turns clockwise. (See Figure 13(a).) Let c_L denote the center of $D_L(s)$. We denote by D_θ the disk obtained by rotating D by an angle θ around c_L . We define θ^* as the smallest $\theta \in (0, 2\pi]$ such that $\theta \leq \pi/3$ and $D_\theta \setminus D$ touches ∂P , or $\theta > \pi/3$ and D_θ touches ∂P . Since $D_L(s) \cup D_R(s) \not\subset P$, such a θ^* exists. There is a path from s to q consisting of an arc of $D_L(s)$, an arc of D_{θ^*} , a line segment, an arc of D and C_3 . (See Figure 13(b).) This means that t can be reached by a canonical LR-start and a right-turning path.

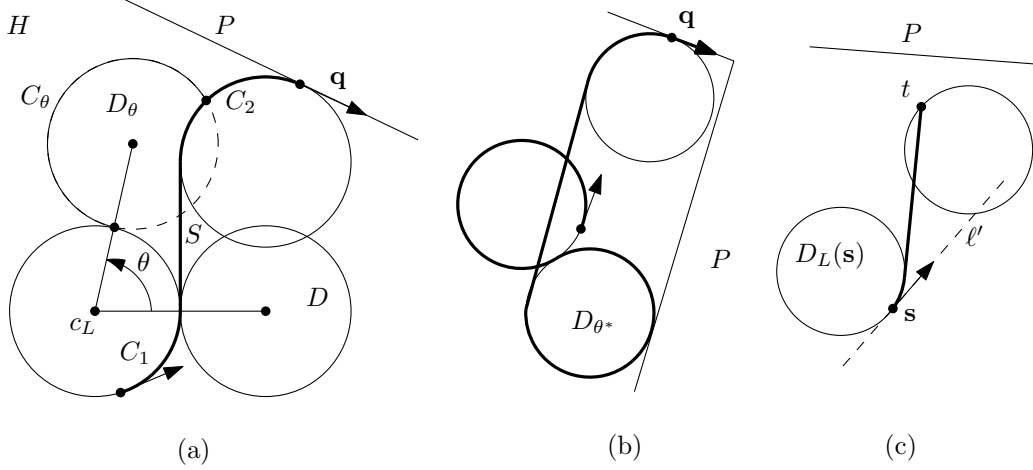


Figure 14: Proof of Proposition 12, case (ii).

Case (ii). We assume that $\gamma_1 = C_1SC_2$, where C_1 is left-turning, S is a segment and C_2 is right-turning. (See Figure 14.)

Let us first assume that $m = 1$, that is that $\gamma = \gamma_1 = C_1SC_2$ and that C_2 has length less than π . In this case, t lies in $\text{DA}(s)$. Indeed, if C_1 has length larger than π or if t lies to the left of the directed line ℓ' defined by s , then $t \in \text{LDA}(s)$. (See Figure 14(c)). If C_1 has length less than π and t lies to the right of ℓ' , then $t \in \text{RDA}(s)$, since γ cannot enter $D_R(s)$ because C_2 has length less than π .

It remains to consider the case where $m \geq 2$ or the length of C_2 is at least π . Let D be the unit disk tangent to $D_L(s)$ and lying to the right of the segment S , and denote by D_θ the disk obtained by rotating D by an angle $\theta \in (0, 2\pi]$ counterclockwise around c_L . (See Figure 14(a).) Let C_θ be the clockwise arc of D_θ starting at $D_L(s) \cap D_\theta$ and going clockwise until the first intersection point with the path SC_2 , or returning to its starting point if there is no such intersection. Let θ^* be the smallest value of $\theta \in (0, 2\pi]$ such that C_θ touches ∂P . Since $D_L(s) \cup D_R(s) \not\subset P$, such a θ^* exists. Now q can be reached by a path consisting of an arc of $D_L(s)$, an arc of D_{θ^*} , a line segment, and an portion of C_2 . (See Figure 14(b).) It follows that t can be reached by a canonical LR-start and a right-turning path.

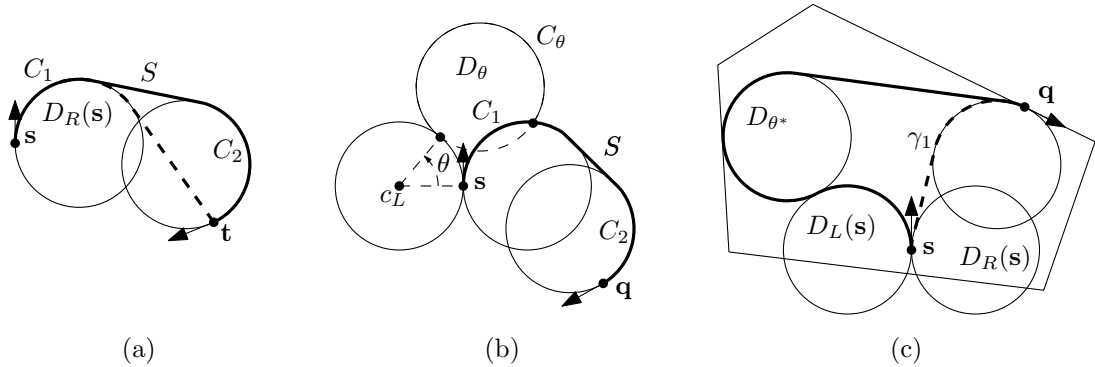


Figure 15: Proof of Proposition 12, case (iii).

Case (iii). We assume that $\gamma_1 = C_1SC_2$, where C_1 and C_2 are right-turning and S is a segment. If the length of C_2 is less than π and $m = 1$ (see Figure 15(a)), then $t \notin D_R(\mathbf{s})$, and therefore $t \in \text{RDA}(\mathbf{s})$. We therefore assume that $m \geq 2$ or the length of C_2 is at least π .

We denote by D_θ the disk obtained by rotating $D_R(\mathbf{s})$ around the center c_L of $D_L(\mathbf{s})$ by an angle $\theta \in (0, \pi/2]$. (See Figure 15(b).) When D_θ intersects γ_1 , we denote by C_θ the arc of ∂D_θ that starts at $D_L(\mathbf{s}) \cap D_\theta$ and goes clockwise until it meets γ_1 . Otherwise, we denote $C_\theta = \partial D_\theta$. As before, let θ^* be the smallest value of $\theta \in (0, 2\pi]$ such that C_θ touches ∂P . Again, θ^* exists since $D_L(\mathbf{s}) \cup D_R(\mathbf{s}) \not\subset P$. Now \mathbf{q} can be reached by a path that consists of an arc of $D_L(\mathbf{s})$, an arc of D_{θ^*} , a segment tangent to D_{θ^*} and γ_1 , and a subpath of γ_1 . (See Figure 15(c).) This implies again that t can be reached by a canonical LR-start followed by a right-turning path. \square

We can give a somewhat different characterization:

Proposition 13. *Let P be a convex polygon, let \mathbf{s} be a starting configuration in P , and let $t \in P$ be reachable from \mathbf{s} by a bounded-curvature path. Then t is reachable by a path of the form CCSCS. More precisely, t is reachable by a path of the form CS, or it is reachable by a path of the form CCSCS, where the two final disks touch the boundary of P , and the path goes through these touching points.*

5 Putting everything together

In this section, we show how to construct the reachable region when \mathbf{s} is an arbitrary configuration in P . We obtain it by combining the results in sections 3 and 4. We will prove the following:

Theorem 14. *Let P be an n -sided, convex polygon, and let \mathbf{s} be a configuration inside P . Then the reachable region $\text{REACH}(\mathbf{s})$ from \mathbf{s} inside P is delimited by $O(n)$ arcs of unit circles, and we can compute $\text{REACH}(\mathbf{s})$ in $O(n^2)$ time.*

Proof. Let t be a point in $\text{REACH}(\mathbf{s})$. By Proposition 12, either t is in $\text{DA}(\mathbf{s})$, or it can be reached after a canonical start. We only consider canonical RL-starts; the case of LR-starts can be handled symmetrically.

The directly accessible region $\text{DA}(\mathbf{s})$ is delimited by two circle arcs, which can be computed in $O(n)$ time by brute force. We determine the at most $2n$ canonical RL-starts by brute force, in $O(n^2)$ time. For each canonical start, by Proposition 11, we compute in $O(n)$ time a set of $O(n)$ configurations on the side of P such that the union of their LDAs form the reachable region after this canonical start.

We have thus obtained a set of $O(n^2)$ configuration on the boundary of P such that the union of their LDAs with $\text{DA}(\mathbf{s})$ is $\text{REACH}(\mathbf{s})$. By Lemma 10, we only need to keep two such configurations per edge: the first and the last one. As we have only $O(n)$ arcs to consider, we can construct $\text{REACH}(\mathbf{s})$ by inserting these arcs one by one, and updating the reachable region by brute force. As these arcs are arcs of unit circles, each one of them appears only once along the boundary of $\text{REACH}(\mathbf{s})$. So overall, it takes $O(n^2)$ time. \square

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